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Hamiltonian Formulation of Systems with Higher Order Derivatives

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Abstract Using the framework of variational principle in mechanics, we obtain a new description of the Hamiltonian formulation of systems with higher order derivatives, involving the introduction of the usual definition of the generalized momenta as presented for the first order systems. The canonical action is obtained in terms of canonical coordinates q_i $(i = 1, 2, \dots, n)$. Besides, the path integral quantization is obtained as an integration over the canonical coordinate q_i without any need to integrate over the higher order derivatives $(\bar{q}_i = \dot{q}_i)$. Two examples are studied.

Keywords Hamiltonian and Lagrangian approach · Higher order Lagrangians · Quantization of field systems

1 Introduction

Systems with higher order Lagrangians have been studied with increasing interest because they appear in many relevant physical problems. The most popular examples being perhaps higher order regularization of quantum gauge field theories and so-called rigid strings [1, 2], rigid particles [3, 4], a relativistic particle with curvature and torsion in three dimensional space-time [5] and in the work of Podolsky's [6] and Bopp [7], who independently proposed generalization of electrodynamics containing second order derivatives. Also the work of Green [8, 9], who proposed a generalized meson-field theory. Besides this, we have other examples, such as, the consistent of ultraviolet divergence in gauge invariant supersymmetric theories [10, 11], or the effective Lagrangian in gauge theories [12, 13].

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The treatment for theories with higher order derivatives has been first developed by Ostrogradskii [14] and leads to obtain the Euler–Lagrange equations and the Hamilton equations of motion.

The path integral quantization of systems with higher order derivatives is studied [15], using Hamilton–Jacobi [HJ] method [16–22], where these systems are converted into first order systems with constraints. Also, important applications in the contest of the [HJ] formalism were made in [23–25], including systems with higher order derivatives. Another work of the [HJ] formalism of systems with higher order derivatives is given in [26, 27], where the action function is obtained for both constrained and unconstrained systems by solving the appropriate set of Hamilton–Jacobi partial differential equations [HJPDEs], and used it to determine the solution of the equations of motion by using the WKB approximation.

Very recently, a new development of systems with higher order fractional derivatives was made in Ref. [28], and the path integral quantization for both conservative and non conservative systems are recovered.

Furthermore, in Refs. [29, 30], the authors studied the Hamiltonian formulation of higher order dynamical systems using Dirac's approach to constrained dynamics, where the Hamiltonian formulation of regular higher order Lagrangians is developed, and the conventional description of such systems due to Ostrogradskii is recovered. Also, a new development for systems with higher order derivatives and degenerate coordinate, was made in Ref. [31].

Other treatments for systems with higher order derivatives are presented in Ref. [32], where the authors have analyzed systems with higher order derivatives, using the discrete variational principle to obtain the discrete Euler–Lagrange equations for higher-order Lagrangians and the corresponding discrete Hamiltonian.

The aim of this paper is to give a new Hamiltonian formulation for systems with higher order derivatives and to show that our Hamiltonian does not depend on any higher order derivatives of the coordinates.

The paper is organized as follows.

In Sect. 2 construction of higher order Lagrangians is reviewed. Section 3 presents the Hamiltonian and the path integral quantization for these systems. Two examples are described in Sect. 4. Conclusions are presented in Sect. 5.

2 The Ostrogradskii's Construction

The Lagrangian formulation of these theories require the configuration space formed by n generalized coordinates $q_i, \dot{q}_i$ and $\ddot{q}_i$. The Euler Lagrangian equations of motion, which are obtained from

$$S = \int L(q_i, \dot{q}_i, \ddot{q}_i)dt, \tag{1}$$

using the Hamilton principle, are given by:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0. \tag{2}$$

The passage from the Lagrangian approach to the Hamiltonian approach is achieved by introducing the generalized momenta (p_i, π_i) conjugated to the generalized coordinates $(q_i, \dot{q}_i)$ respectively as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right), \tag{3}$$

$$\pi_i = \frac{\partial L}{\partial \ddot{q}_i}. \tag{4}$$

The phase-space will then be spanned by the canonical variables (q_i, p_i) and $(\bar{q}_i, \pi_i)$, where $(\bar{q}_i = \dot{q}_i)$, and the Hamiltonian is given by

$$H = p_i \bar{q}_i + \pi_i \ddot{q}_i - L. \tag{5}$$

This means that, the Hamiltonian is a function of the form

$$H = H(q_i, p_i, \bar{q}_i, \pi_i, t). \tag{6}$$

In the generalized case:

$$L = L(q_i, \dot{q}_i, \ddot{q}_i, q_i^{(3)}, \dots, q_i^{(n)}; t). \tag{7}$$

Consider the variational principle, it is clear that the Euler–Lagrange equations of motion of the system are given by

$$\sum_{i=0}^{n} (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial q^{(i)}}\right) = 0, \quad q^{(i)} = \frac{d^i q}{dt^i},\ i = 0, 1, 2, \dots, n. \tag{8}$$

Following Ostrogradskii's [14] and in order to simplify the expression of these equations, let us introduce the generalized momenta (p_i, π_i) conjugated to the generalized coordinates $(q_i, \dot{q}_i)$ respectively as

$$p_i^{k-1} = \frac{\partial L}{\partial q_i^{(k)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(k+1)}}\right), \quad k = 1, 2, 3, \dots, n-1, \tag{9}$$

$$\pi_i = \frac{\partial L}{\partial q_i^{(n)}}. \tag{10}$$

The phase-space will then be spanned by the canonical variables (q_i, p_i^{k-1}) and $(\bar{q}_i^{(k)}, \pi_i)$, where $(\bar{q}_i^{(k)} = q_i^{(k+1)})$, and the Hamiltonian is given by

$$H = \sum_{k=1}^{n-1} p_k q^{(k)} + \pi_i q_i^{(n)} - L. \tag{11}$$

This means that, the Hamiltonian is a function of the form

$$H = H(q_i, p_i^{k-1}, \bar{q}_i^k, \pi_i, t). \tag{12}$$

3 The Hamiltonian Formulation

The dynamics of a physical system is encoded in the Lagrangian, a function of the positions and velocities of all the degrees of freedom which comprise the system [33–35]. To extract the dynamics one considers paths in the configuration space. For a given path, one calculates the position and velocities at each time and also the value of the Lagrangian. The Lagrangian formulation of classical physics requires the configuration space formed by generalized coordinate q and generalized velocities $\dot{q}$ and $\ddot{q}$ defined as:

$$L = L(q, \dot{q}, \ddot{q}; t). \tag{13}$$

Integration of the Lagrangian values against time gives the action, a functional of path in configuration space as,

$$S = \int L(q, \dot{q}, \ddot{q}; t)dt. \tag{14}$$

The evolution of the classical system is then obtained from the Euler–Lagrange equations of motion. These equations follow from the Hamilton's principle, the classical dynamical content of a system is prescribed by the requirement that the time integral of the Lagrangian (the classical action S) be an extremum.

$$\delta S = \delta \int L(q, \dot{q}, \ddot{q}; t)dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} + \frac{\partial L}{\partial t}\delta t \right). \tag{15}$$

In deriving (15), it was assumed that $\dot{q}$ and $\ddot{q}$ are not independent of q, so that $\delta\dot{q} = \frac{d}{dt}\delta q$, $\delta\ddot{q} = \frac{d^2}{dt^2}\delta q$. This corresponds to the variational principle in configuration space.

Imposing $\delta S = 0$, we obtain the Euler–Lagrange equations of motion

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0. \tag{16}$$

The Hamiltonian formulations defined as

$$H = p_1\dot{q} + p_2\ddot{q} - L. \tag{17}$$

Calculating the total differential of this Hamiltonian we obtain

$$dH = p_1 d\dot{q} + \dot{q}dp_1 + p_2 d\ddot{q} + \ddot{q}dp_2 - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial \ddot{q}}d\ddot{q} - \frac{\partial L}{\partial t}dt. \tag{18}$$

The generalized momenta p_1 and p_2 corresponding to $\dot{q}$ and $\ddot{q}$ can be defined as follows

$$p_1 = \frac{\partial L}{\partial \dot{q}}, \tag{19}$$

$$p_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{20}$$

Substituting the values of momenta (19), (20) into equation (18) we have:

$$dH = \dot{q}dp_1 + \ddot{q}dp_2 - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt. \tag{21}$$

Making use of the Euler–Lagrange equation (16), we get

$$dH = \dot{q}dp_1 + \ddot{q}dp_2 + \left(-\frac{dp_1}{dt} + \frac{d^2 p_2}{dt^2}\right)dq - \frac{\partial L}{\partial t}dt. \tag{22}$$

This means that, the Hamiltonian is a function of the form

$$H = H(q, p_1, p_2, t), \tag{23}$$

and the total differential of this function reads as:

$$dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p_1}dp_1 + \frac{\partial H}{\partial p_2}dp_2 + \frac{\partial H}{\partial t}dt. \tag{24}$$

Comparing equations (22) and (24), we get the following Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p_1}, \qquad \ddot{q} = \frac{\partial H}{\partial p_2}, \tag{25}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{26}$$

$$\frac{\partial H}{\partial q} = -\dot{p}_1 + \ddot{p}_2. \tag{27}$$

It worthwhile to mention that the results (25–27) are the same as those obtained in Ref. [36] for the limiting case of integer orders of fractional equations of motion.

In the generalized case:

$$L = L(q, \dot{q}, \ddot{q}, q^{(3)}, \dots, q^{(n)}; t). \tag{28}$$

Integration of the Lagrangian values against time gives the action, a functional of path in configuration space as,

$$S = \int L(q, \dot{q}, \ddot{q}, q^{(3)}, \dots, q^{(n)}; t)dt. \tag{29}$$

The evolution of the classical system is then obtained from the Euler–Lagrange equations of motion. So, we obtain the Euler–Lagrange equations of motion

$$\sum_{i=0}^{n}(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial q^{(i)}}\right) = 0, \quad q^{(i)} = \frac{d^i q}{dt^i}, \; i = 0, 1, 2, \dots, n. \tag{30}$$

The Hamiltonian formulation defined as

$$H = \sum_{k=1}^{n} p_k q^{(k)} - L, \quad k = 1, 2, 3, \dots, n. \tag{31}$$

Calculating the total differential of this Hamiltonian we obtain

$$dH = \sum_{k=1}^{n}\left[p_k dq^{(k)} + q^{(k)}dp_k - \frac{\partial L}{\partial q^{(k)}}dq^{(k)}\right] - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt. \tag{32}$$

The generalized momenta p_k corresponding to $q^{(k)}$ can be defined as follows

$$p_k = \frac{\partial L}{\partial q^{(k)}}. \tag{33}$$

Substituting the values of momenta into (32) we have:

$$dH = \sum_{k=1}^{n} q^{(k)} dp_k - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt. \tag{34}$$

Making use of the Euler–Lagrange equation (30), we get

$$-\frac{\partial L}{\partial q} = \sum_{k=1}^{n} (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q^{(k)}}\right) = \sum_{k=1}^{n} (-1)^k \frac{d^k p_k}{dt^k}. \tag{35}$$

Substituting (35) into (34), we get

$$dH = \sum_{k=1}^{n} \left[q^{(k)} dp_k + (-1)^k \frac{d^k p_k}{dt^k} dq \right] - \frac{\partial L}{\partial t} dt. \tag{36}$$

This means that, the Hamiltonian is a function of the form

$$H = H(q, p_k, t), \quad k = 1, 2, 3, \ldots, n, \tag{37}$$

and the total differential of this function reads as:

$$dH = \frac{\partial H}{\partial q} dq + \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt. \tag{38}$$

Comparing (36) and (38), we get the following Hamilton's equations of motion

$$q^{(k)} = \frac{\partial H}{\partial p_k}, \tag{39}$$

$$\frac{\partial H}{\partial q} = \sum_{k=1}^{n} (-1)^k p_k^{(k)}. \tag{40}$$

In this case, the path integral quantization is given by

$$K = \int dq dp_k e^{i \int (\sum_{k=1}^{n} p_k q^{(k)} - H(q, p_k, t)) dt}. \tag{41}$$

For quadratic $H(q, p_k, t)$ in terms of p_k, and after integration over the momenta p_k we obtain

$$K = \int dq e^{i \int L dt}. \tag{42}$$

One should notice that the path integral quantization is obtained as an integration over the canonical coordinates q_i without any need to integrate over higher order derivatives $(q_i, \bar{q}_i)$ as given by Ostrogradskii formulation.

4 Examples

As a first example, let us consider the Lagrangian [29]

$$L = \frac{1}{2}ax\ddot{x}^2 - \frac{1}{2}bx\dot{x}^2. \tag{43}$$

The momenta p_1 and p_2 are given as

$$p_1 = \frac{\partial L}{\partial \dot{x}} = -bx\dot{x}, \tag{44}$$

$$p_2 = \frac{\partial L}{\partial \ddot{x}} = ax\ddot{x}. \tag{45}$$

Making use of (17), the Hamiltonian becomes

$$H = p_1\dot{q} + p_2\ddot{q} - L = -\frac{p_1^2}{2bx} + \frac{p_2^2}{2ax}, \tag{46}$$

and the Hamilton's equations of motions are

$$\dot{x} = -\frac{p_1}{bx}, \tag{47}$$

$$\ddot{x} = \frac{p_2}{ax}, \tag{48}$$

$$\frac{\partial H}{\partial x} = -\dot{p}_1 + \ddot{p}_2. \tag{49}$$

Equation (49) gives:

$$\frac{3}{2}a\ddot{x}^2 + bx\ddot{x} + 2a\dot{x}x^{(3)} + \frac{1}{2}b\dot{x}^2 + axx^{(4)} = 0. \tag{50}$$

Besides, the path integral is given by

$$K = \int dx dp_1 dp_2 e^{i\int (p_1\dot{x} + p_2\ddot{x} + \frac{p_1^2}{2bx} - \frac{p_2^2}{2ax})dt}. \tag{51}$$

Integrating over the momenta p_1 and p_2 we arrive at the result

$$K = \int dx e^{i\int (\frac{1}{2}ax\ddot{x}^2 - \frac{1}{2}bx\dot{x}^2)dt}. \tag{52}$$

The path integral (52), is an integration over the canonical coordinate x, without any need to any integration over the velocity $\bar{x} = \dot{x}$ as given by Ostrogradskii formulation.

As a second example, we consider the singular Lagrangian [27]

$$L = \frac{1}{2}(\ddot{q}^2 - \dot{q}^2). \tag{53}$$

The momenta p_1 and p_2 are given as

$$p_1 = \frac{\partial L}{\partial \dot{q}} = -\dot{q}, \tag{54}$$

$$p_2 = \frac{\partial L}{\partial \ddot{q}} = \ddot{q}. \tag{55}$$

Making use of (17), the Hamiltonian becomes

$$H = p_1 \dot{q} + p_2 \ddot{q} - L = -\frac{1}{2} p_1^2 + \frac{1}{2} p_2^2, \tag{56}$$

and the Hamilton's equations of motions are

$$\dot{q} = -p_1, \tag{57}$$

$$\ddot{q} = p_2, \tag{58}$$

$$\frac{\partial H}{\partial q} = 0 = -\dot{p}_1 + \ddot{p}_2. \tag{59}$$

Equation (59) gives:

$$q^{(4)} + \ddot{q} = 0, \tag{60}$$

which has the following solution

$$q = c_1 + c_2 t + c_3 \cos t + c_4 \sin t, \tag{61}$$

where c_1, c_2, c_3 and c_4 are constants. The result obtained in (61) is the same as obtained in Ref. [27].

The path integral is given by

$$K = \int dq e^{i \int (\frac{1}{2}(\ddot{q}^2 - \dot{q}^2)) dt}. \tag{62}$$

5 Conclusion

In this paper we have studied the Hamiltonian formulation of systems with higher order derivatives. We have investigated two systems, in the first one, the path integral quantization is obtained as an integration over the canonical coordinate x without any need to integrate over the velocity $\bar{x} = \dot{x}$ as given by Ostrogradskii formalism. In the second example, the path integral is obtained directly as an integration over the canonical coordinate q without any need to integrate over the variable $\dot{q}$.

We observed that the classical equations of motion obtained in this work are in complete agreement with those obtained by the Lagrangian formulation. The advantages of using our formalism is that, the path integral quantization is obtained directly over the space q_i without any need to introduce extra coordinates $\bar{q}_i$ as given by Ostrogradskii formalism.

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References

1. Polyakov, A.: Nucl. Phys. B **268**, 406 (1986)
2. Kleinert, H.: Phys. Lett. B **174**, 335 (1986)
3. Govaerts, J.: Phys. Lett. B **293**, 327 (1992)

4. Govaerts, J.: Z. Phys. C **57**, 59 (1993)
5. Nesterenko, V.: Phys. Lett. B **327**, 50 (1994)
6. Podolsky, B., Schwed, P.: Rev. Mod. Phys. **20**, 40 (1948)
7. Bopp, F.: Ann. Phys. **38**, 345 (1940)
8. Green, A.: Phys. Rev. **73**, 26 (1948)
9. Green, A.: Phys. Rev. **75**, 1926 (1949)
10. Stelle, K.S.: Phys. Rev. D **16**, 953 (1977)
11. Stelle, K.S.: Nucl. Phys. B **268**, 113 (1986)
12. Alekseev, A.I., Arbuzov, B.A., Baikov, B.A.: Theor. Math. Phys. **52**, 739 (1982)
13. Alekseev, A.I., Arbuzov, B.A., Baikov, B.A.: Theor. Math. Phys. **59**, 372 (1983)
14. Ostrogradskii, M.: Mem. Acad. St. Petersbourg **VI**, 385 (1850)
15. Shihada, O., Muslih, S., Farahat, N., Sadallah, M.: A treatment of regular systems with higher order Lagrangians as singular systems with first order Lagrangians. In: Muslih, S., Okasha, M. (eds.) The First International Conference on Mathematical Sciences, Al-Azhar University-Gaza, 15–17 May 2006
16. Güler, Y.: Nuovo Cimento B **107**, 1143 (1992)
17. Güler, Y.: Nuovo Cimento B **107**, 1389 (1992)
18. Muslih, S., Güler, Y.: Nuovo Cimento B **110**, 307 (1995)
19. Rabei, E., Güler, Y.: Phys. Rev. A **46**, 3513 (1992)
20. Muslih, S.I.: Gen. Relativ. Gravit. **34**, 1059 (2002)
21. Muslih, S.: Mod. Phys. Lett. A **19**, 863 (2004)
22. Baleanu, D.: Central Eur. J. Phys. **4**, 508 (2006)
23. Muslih, S.: Mod. Phys. Lett. A **17**, 2383 (2002)
24. Pimentel, B.M., Teixeira, R.G.: Nuovo Cimento B **11**, 841 (1998)
25. Pimentel, B.M., Teixeira, R.G.: Nuovo Cimento B **113**, 805 (1998)
26. Hasan, E.H., Rabei, E.M., Ghassib, H.B.: Int. J. Theor. Phys. **43**(11), 2285 (2004)
27. Rabei, E.M., Tarawneh, M.M.: Int. J. Theor. Phys. **46**, 884 (2007)
28. Baleanu, D., Muslih, S., Tas, K.: J. Math. Phys. **47**, 103503 (2006)
29. Govaerts, J., Rashid, M.: arXiv:hep-th/9403009 (1994)
30. Rashid, M., Khalil, S.: Int. J. Mod. Phys. A **11**, 4551 (1996)
31. Gitman, D.M., Tyutin, I.V.: Nucl. Phys. B **630**, 509 (2002)
32. Baleanu, D., Jarad, F.: Nuovo Cimento B **120**, 931 (2005)
33. Goldstein, H.: Classical Mechanics, 2nd edn. Addison-Wesley, Reading (1980)
34. Govaerts, J.: Hamiltonian Quantisation and Constrained Dynamics, vol. 4. Leuven University Press, Leuven (1991)
35. Gitman, D.M., Tyutin, I.V.: Quantization of Fields with Constraints. Springer, Berlin (1990)
36. Rabei, E.M., Nawafleh, K.I., Hijjawi, R.S., Muslih, S.I., Baleanu, D.: J. Math. Anal. Appl. **327**, 891 (2007)